General Relativity $\boxed{\text{Week 3}}$

Last time:

If $p \in M$, tensors of type (k,l) over p:

$$T: \underbrace{T_p^*M \times T_p^*M \times \dots \times T_p^*M}_{k} \times \underbrace{T_pM \times \dots \times T_pM}_{l} \longrightarrow \mathbb{R}$$

In local coordinates:

$$T = T^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_l} \frac{\partial}{\partial x^{\alpha_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\alpha_k}} \otimes dx^{\beta_1} \otimes \dots \otimes dx^{\beta_l}$$

So: $T^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_l} = T\left(dx^{\alpha_1}, \dots, dx^{\alpha_k}; \frac{\partial}{\partial x^{\beta_1}}, \dots, \frac{\partial}{\partial x^{\beta_l}}\right)$

Operations:

- Tensor product: $\left(\underset{(k_1,l_1)}{T_1} \otimes \underset{(k_2,l_2)}{T_2}\right)^{\alpha_1 \dots \alpha_{k_1+k_2}}_{\beta_1 \dots \beta_{l_1+l_2}} = \left(T_1\right)^{\alpha_1 \dots \alpha_{k_1}}_{\beta_1 \dots \beta_{l_1}} \cdot \left(T_2\right)^{\alpha_{k_1+1} \dots \alpha_{k_1+k_2}}_{\beta_{l_1+1} \dots \beta_{l_1+l_2}}$

- Contraction (tr): With respect to one contravariant and one covariant index

  Ex: 1^{st} - 1^{st} indices:

$$\operatorname{tr} T^{i_1 \dots i_{k-1}}_{j_1 \dots j_{l-1}} = T^{a\, i_1 \dots i_{k-1}}_{a\, j_1 \dots j_{l-1}}$$

Coordinate independent! Like the trace of a linear endomorphism

In fact: A linear map $L: T_pM \longrightarrow T_pM$

can be viewed as a $(1,1)$ tensor $\tilde{L}: T_p^*M \times T_pM \longrightarrow \mathbb{R}$

So that $\tilde{L}(\omega, v) = \omega\big(L(v)\big) \iff L(v) = \tilde{L}(\cdot, v)$

In general: $\tilde{L}: \underbrace{T_p^*M \times \dots \times T_p^*M}_{k} \times \underbrace{T_pM \times \dots \times T_pM}_{l} \longrightarrow \mathbb{R}$

can be viewed as $L: \underbrace{T_p^*M \times \dots \times T_p^*M}_{k-k_1} \times \underbrace{T_pM \times \dots \times T_pM}_{l-l_1} \longrightarrow \otimes^{k_1} T_pM \otimes^{l_1} T_p^*M$

So: If L is $(1,1)$: $\operatorname{tr} L$ is the usual trace.

Examples: If $\omega \in T_p^* M$, $v \in T_p M$:

$$(\omega \otimes v)^{\alpha}_{\beta} = \omega_\beta v^\alpha$$

$$\operatorname{tr}(\omega \otimes v) = \omega_\alpha v^\alpha = \omega(v)$$

$$\operatorname{tr}(\operatorname{tr}(g \otimes X \otimes Y)) = g(X,Y)$$

metric vectors

Musical isomorphisms:

Let g be a non-degenerate symmetric $(0,2)$ tensor
(e.g. coming from Lorentzian inner product)
It defines a natural isomorphism between T_pM and T_p^*M
by $X \longrightarrow \omega = g(X, \cdot)$

In coordinates: $\omega_\alpha = g_{\alpha\beta} X^\beta$ ← Musical isomorphisms

$$\Leftrightarrow X^\alpha = g^{\alpha\beta} \omega_\beta$$

↑ Inverse matrix of $[g]$!

More generally, I can raise - lower indices with g (i.e. changing the type of tensors). I will keep the same symbol for the new tensor

e.g. $T_{\alpha\beta} = g_{\alpha\gamma} T^{\gamma}_{\beta}$

I can extend g to inner product on tensors by

$$g(F,G) = g_{a_1 i_1} \cdots g_{a_k i_k} g^{\beta_1 j_1} \cdots g^{\beta_l j_l} F^{a_1 \cdots a_k}_{\beta_1 \cdots \beta_l} G^{i_1 \cdots i_k}_{j_1 \cdots j_l}$$

e.g. if ω_1, ω_2 are covectors:

$$g(\omega_1, \omega_2) = g^{\alpha\beta} (\omega_1)_\alpha (\omega_2)_\beta$$

The musical isomorphisms are <u>isometries</u> with respect to this (extended) inner product.

<u>Tensor Fields</u>

<u>Def</u>: T is a <u>(smooth) tensor field</u> of type (k,l) if $\forall p \in M$, it is an assignment $p \longrightarrow T|_p \in \otimes^k T_pM \otimes^l T_p^*M$ with smooth components in any local coordinate chart.

<u>Ex</u>: Lorentzian metric g is a $(0,2)$ tensor field.

<u>Note</u>: if $\omega_1, \ldots, \omega_k \in \Gamma^*(M)$, $v_1, \ldots, v_l \in \Gamma(M)$

$$T(\omega_1, \ldots, \omega_k, v_1, \ldots, v_l)|_p \text{ depends only on } \omega_i|_p, \ldots$$

So $T: \Gamma^*(M) \times \cdots \times \Gamma^*(M) \times \Gamma(M) \times \cdots \times \Gamma(M) \longrightarrow C^\infty(M)$

is <u>$C^\infty(M)$ - multilinear</u> !

$$T(f \cdot \omega_1, \ldots) = f \cdot T(\omega_1, \ldots)$$

<u>Theorem</u>: A map $T: \Gamma^*(M) \times \cdots \times \Gamma^*(M) \times \Gamma(M) \cdots \times \Gamma(M) \longrightarrow C^\infty(M)$ is a tensor field if and only if it is $C^\infty(M)$ - multilinear

Ex: On $\mathbb{R}^n$: $\operatorname{div}: \Gamma(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$

is not a tensor field.

Lie brackets and flows:

Let $X \in \Gamma(M)$. It is a first order differential operator

$X: C^\infty(M) \to C^\infty(M)$. In local coordinates: $X = X^a \frac{\partial}{\partial x^a}$

For $X, Y \in \Gamma(M)$: The composition $X \cdot Y$ is a 2^{nd} order diff. operator

But $[X,Y] = X\cdot Y - Y \cdot X$ is 1^{st} order

In local coordinates:

$$[X,Y]^a = X^b \frac{\partial}{\partial x^b} Y^a - Y^b \frac{\partial}{\partial x^b} X^a$$

Commutator or Lie-bracket

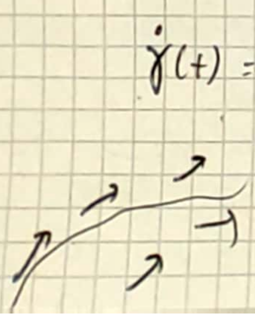
Properties:
1) $[X,Y] = -[Y,X]$ (anticommutativity)
2) $\mathbb{R}$-bilinear
3) Jacobi's identity: $[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0$

So $(\Gamma(M), [\cdot,\cdot])$ forms a Lie algebra.

Of which group?

Def: Let $X \in \Gamma(M)$. Integral curve of X: $\gamma: (a,b) \to M$,

$$\dot\gamma(t) = X|_{\gamma(t)} \quad \forall t \in (a,b)$$



In local coordinates: $\dot{\gamma}^{\mu}(t) = X^{\mu}(\gamma(t))$

$\uparrow$ 1^{st} order ODE

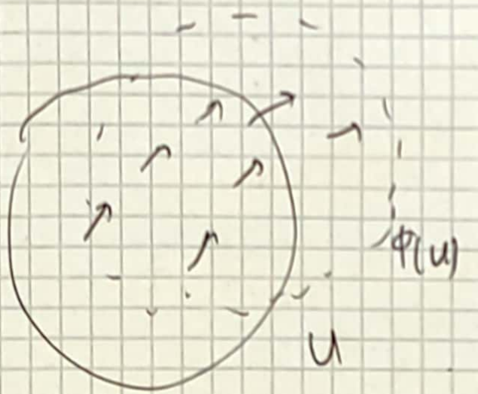
Fundamental theorem of ODEs:

The initial value problem $\begin{cases} \dot{y} = F(t,y) \\ y(0) = y_0 \end{cases}$ has a unique solution (as long as F is regular enough) depending smoothly on the initial data.

<u>Theorem</u>: Let $X \in \Gamma(M)$. $\forall\, U \subseteq M$ open such that $\bar{U} \subset\subset M$ (i.e. U is pre-compact), $\exists\, \delta > 0$ and a unique flow map $\Phi : (-\delta, \delta) \times U \to M$

smooth

t x



associated to X, i.e.

1) $\Phi(0, x) = x$

2) $\frac{\partial}{\partial t} \Phi(t,x) = X(\Phi(t,x))$

So Φ: Moves U along the integral curve of X

<u>Note</u>: If X is compactly supported: $\delta = +\infty$, Φ extends to $(-\infty, +\infty) \times M$

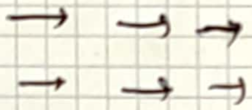
We will denote $\Phi(t, \cdot)$ by $\Phi_t(\cdot)$

$\{\Phi_t\}_{t \in \mathbb{R}}$: Flow of local diffeomorphisms generated by X

1-parameter group: $\Phi_t \circ \Phi_s = \Phi_{t+s}$

$\mathrm{Diff}_{com}(M)$: Has Lie algebra $\Gamma_{com}(M)$

<u>Ex</u>: • If $X = \frac{\partial}{\partial x^1}$: $\Phi_t(x^1, x^2, \dots, x^n) = (x^1 + t, x^2, \dots, x^n)$

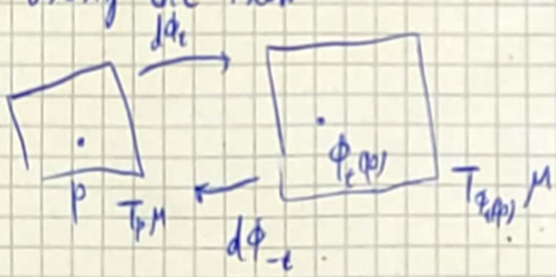


• On $\mathbb{R}$: $X = y^2 \frac{d}{dy}$ has finite time of existence

$$(\dot{y} = y^2)$$

<u>Lie derivative</u>

Derivative along the flow



<u>Definition</u>

1) If $f \in C^\infty(M)$: $\mathcal{L}_X f(p) = \lim_{t \to 0} \dfrac{f(\phi_t(p)) - f(p)}{t}$

2) If $Y \in \Gamma(M)$: $\mathcal{L}_X Y|_p = \lim_{t \to 0} \dfrac{d\phi_{-t}(Y|_{\phi_t(p)}) - Y|_p}{t}$

3) If $\omega \in \Gamma^*(M)$: $\mathcal{L}_X \omega|_p = \lim_{t \to 0} \dfrac{\overbrace{\phi_t^*}^{\text{pull-back}}(\omega|_{\phi_t(p)}) - \omega|_p}{t}$

4) For higher order tensors

Extend by linearity and the condition that

$$\mathcal{L}_X(F \otimes G) = (\mathcal{L}_X F) \otimes G + F \otimes (\mathcal{L}_X G)$$

<u>Rmk</u>: This way: $\mathcal{L}_X$ commutes with contractions

Theorem: 1) $\mathcal{L}_X f = X(f)$

2) $\mathcal{L}_X Y = [X,Y]$

Proof: • At a point where $X|_p \neq 0$: choose local coordinates where $X = \frac{\partial}{\partial x^1}$

Then φ_t: translation in the x^1 variable

Remark: Since we want to prove a coordinate-independent statement, it suffices to show it in one coordinate system

From the definition:

$$\mathcal{L}_X f = \frac{\partial f}{\partial x^1}$$

$$(\mathcal{L}_X Y)^k = \frac{\partial Y^k}{\partial x^1}$$

• At a point p where $X \equiv 0$ on a neighborhood of p:

$\varphi_t \equiv id$ there $\Rightarrow \mathcal{L}_X f = 0$, $\mathcal{L}_X Y = 0$

• Any other point: Lies at the closure of the previous regions, so by continuity we get the result $\boxtimes$

If g is a $(0,2)$ tensor:

$$\mathcal{L}_X g = \lim_{t \to 0} \frac{\varphi_t^* g - g}{t}$$

So if g is a Lor. metric:

$$\mathcal{L}_X g = 0 \iff \varphi_t^{(X)} \text{ is an isometry}$$

Such an X: Killing field

Note: $\underbrace{X(g(Y,Z))}_{tr(tr(g\otimes X\otimes Y))} = \cancelto{0}{(\mathcal{L}_X g)}(Y,Z) + g(\mathcal{L}_X Y, Z) + g(Y, \mathcal{L}_X Z)$

- If X generates a group of <u>conformal</u> isometries

$$\mathcal{L}_X g = h \cdot g \quad \text{for some } h \in C^\infty(M)$$

<u>Example</u>: On Minkowski spacetime $(\mathbb{R}^{n+1}, \eta)$.

Killing vector fields:

- Translations $\frac{\partial}{\partial x^a}$, $a = 0, \dots, n$
- Rotations $\Omega_{ij} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$, $i = 1, \dots, n$
- Boosts $\Omega_{0i} = x^0 \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial x^0}$

"Hyperbolic Rotation"

~~Conformal Killing fields on Minkowski~~: $(n \geq 2)$